

NON-RATIONALITY OF THE SYMMETRIC SEXTIC FANO THREEFOLD

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INTRODUCTION

The symmetric sextic Fano threefold is the subvariety X of \mathbb{P}^6 defined by the equations:

$$\sum X_i = \sum X_i^2 = \sum X_i^3 = 0.$$

It is a smooth complete intersection of a quadric and a cubic in \mathbb{P}^5 , with an action of \mathfrak{S}_7 . We will prove that it is not rational.

Any smooth complete intersection of a quadric and a cubic in \mathbb{P}^5 is unirational [E]. It is known that a *general* such intersection is not rational: this is proved in [B] (thm. 5.6) using the intermediate Jacobian, and in [Pu] using the group of birational automorphisms. But neither of these methods allows to prove the non-rationality of any particular such threefold.

Our motivation comes from the recent paper of Prokhorov [P], which classifies the simple finite subgroups of the Cremona group $\text{Cr}_3 = \text{Bir}(\mathbb{P}^3)$. In view of this work our result implies that the alternating group \mathfrak{A}_7 admits only one embedding into Cr_3 up to conjugacy.

To prove our result we use the Clemens-Griffiths criterion ([C-G], Cor. 3.26): if X is rational, its intermediate Jacobian JX is the Jacobian of a curve, or a product of such Jacobians. The presence of the automorphism group \mathfrak{S}_7 , together with the celebrated bound $\#\text{Aut}(C) \leq 84(g-1)$ for a curve C of genus g , immediately implies that JX is not isomorphic to the Jacobian of a curve. To rule out products of Jacobians we need some more information, which is provided by a simple analysis of the representation of \mathfrak{S}_7 on the tangent space $T_0(JX)$.

THE RESULT

Theorem. *The intermediate Jacobian JX is not isomorphic to a Jacobian or a product of Jacobians. As a consequence, X is not rational.*

That the second assertion follows from the first is the Clemens-Griffiths criterion mentioned in the introduction. Since the Jacobians and their products form a closed subvariety of the moduli space of principally polarized abelian varieties, this gives an easy proof of the fact that a general intersection of a quadric and a cubic in \mathbb{P}^5 is not rational.

As mentioned in the introduction, the classification in [P] together with the theorem implies:

Corollary. *Up to conjugacy, there is only one embedding of \mathfrak{A}_7 into the Cremona group Cr_3 , given by an embedding $\mathfrak{A}_7 \subset \text{PGL}_4(\mathbb{C})$.* ■

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(The embedding $\mathfrak{A}_7 \subset \mathrm{PGL}_4(\mathbb{C})$ is the composition of the standard representation $\mathfrak{A}_7 \rightarrow \mathrm{SO}_6(\mathbb{C})$ and the double covering $\mathrm{SO}_6(\mathbb{C}) \rightarrow \mathrm{PGL}_4(\mathbb{C})$.)

The intermediate Jacobian JX has dimension 20. The group \mathfrak{S}_7 acts on JX and therefore on the tangent space $T_0(JX)$; we will first determine this action.

Lemma. *As a \mathfrak{S}_7 -module $T_0(JX)$ is the sum of two irreducible representations, of dimensions 6 and 14.*

Proof : Let V be the standard (6-dimensional) representation of \mathfrak{S}_7 , and $\mathbb{P} := \mathbb{P}(V)$; we will view X as a subvariety of \mathbb{P} , stable under \mathfrak{S}_7 .

Recall that $T_0(JX)$ is $H^1(X, \Omega_X^2)$, and that the exterior product $\Omega_X^1 \otimes \Omega_X^2 \rightarrow K_X$ induces a canonical isomorphism $\Omega_X^2 \xrightarrow{\sim} T_X(-1)$. The exact sequence

$$0 \rightarrow T_X \rightarrow T_{\mathbb{P}|X} \rightarrow \mathcal{O}_X(2) \oplus \mathcal{O}_X(3) \rightarrow 0$$

twisted by $\mathcal{O}_X(-1)$, gives a cohomology exact sequence

$$0 \rightarrow H^0(X, T_{\mathbb{P}}(-1)|_X) \rightarrow H^0(X, \mathcal{O}_X(1)) \oplus H^0(X, \mathcal{O}_X(2)) \rightarrow H^1(X, T_X(-1)) \rightarrow H^1(X, T_{\mathbb{P}}(-1)|_X).$$

From the Euler exact sequence $0 \rightarrow \mathcal{O}_X \rightarrow \mathcal{O}_X(1) \otimes_{\mathbb{C}} V \rightarrow T_{\mathbb{P}|X} \rightarrow 0$ we deduce $H^1(X, T_{\mathbb{P}}(-1)|_X) = 0$ and an isomorphism $V \xrightarrow{\sim} H^0(X, T_{\mathbb{P}}(-1)|_X)$. Thus we find an exact sequence

$$0 \rightarrow V \rightarrow H^0(X, \mathcal{O}_X(1)) \oplus H^0(X, \mathcal{O}_X(2)) \rightarrow T_0(JX) \rightarrow 0,$$

which is equivariant with respect to the action of \mathfrak{S}_7 . As \mathfrak{S}_7 -modules $H^0(X, \mathcal{O}_X(1))$ is isomorphic to V , and $H^0(X, \mathcal{O}_X(2))$ to $S^2V/\mathbb{C}.q$, where q corresponds to the quadric containing X . On the other hand we have $S^2V = \mathbb{C} \oplus V \oplus V_{(5,2)}$, where $V_{(5,2)}$ is the irreducible representation of \mathfrak{S}_7 corresponding to the partition $(5, 2)$ of 7 ([F-H], exercise 4.19). Thus we get $T_0(JX) \cong V \oplus V_{(5,2)}$. ■

Proof of the theorem : We first observe that \mathfrak{S}_7 cannot act faithfully on the Jacobian JC of a curve of genus $g \leq 20$. Indeed by the Torelli theorem, the map $\mathrm{Aut}(C) \rightarrow \mathrm{Aut}(JC)$ is injective and its image has index 1 (if C is hyperelliptic) or 2 otherwise. Thus we find $\#\mathrm{Aut}(C) \geq \frac{1}{2}7! = 2520$. On the other hand we have $\#\mathrm{Aut}(C) \leq 84(g-1) \leq 1596$, a contradiction.

Let A be a principally polarized abelian variety. Recall that A can be written in a unique way as a product $A_1^{a_1} \times \dots \times A_p^{a_p}$, where A_1, \dots, A_p are indecomposable, non isomorphic principally polarized abelian varieties. (this decomposition corresponds to the decomposition of the Theta divisor into irreducible components, see [C-G], Cor. 3.23). Therefore we have

$$\mathrm{Aut}(A) \cong \mathrm{Aut}(A_1^{a_1}) \times \dots \times \mathrm{Aut}(A_p^{a_p}) \quad \text{and} \quad \mathrm{Aut}(A_i^{a_i}) \cong \mathrm{Aut}(A_i)^{a_i} \rtimes \mathfrak{S}_{a_i}.$$

Let $G \subset \mathrm{Aut}(A)$; the product decomposition of A induces a decomposition of G -modules

$$T_0(A) = T_0(A_1^{a_1}) \oplus \dots \oplus T_0(A_p^{a_p}).$$

Fix an integer i ; the group G permutes the factors of $A_i^{a_i}$. If O_1, \dots, O_k are the orbits of G in this action, we have a further decomposition of G -modules

$$T_0(A_i^{a_i}) = T_0(A_i^{O_1}) \oplus \dots \oplus T_0(A_i^{O_k}).$$

Assume that JX is isomorphic to a product of Jacobians $A_1^{a_1} \times \dots \times A_p^{a_p}$, with $A_i \not\cong A_j$ for $i \neq j$. If \mathfrak{S}_7 acts on a set with ≤ 20 elements, its orbits have order 1 or 7 ([D-M], thm. 5.2.B). An orbit with one element means that \mathfrak{S}_7 acts on the Jacobian A_i ; by the lemma this action is faithful, contradicting the beginning of the proof. Thus each a_i must be divisible by 7, which is impossible since $\sum a_i \dim(A_i) = 20$. ■

Remark. The same kind of argument gives a simple proof that the Klein cubic threefold, defined by $\sum_{i \in \mathbb{Z}/5} X_i^2 X_{i+1} = 0$ in \mathbb{P}^4 , is not rational (and by the same token that the general cubic threefold is not rational). The automorphism group of the Klein cubic is $\mathrm{SL}_2(\mathbb{F}_{11})$, of order 660, while its intermediate Jacobian has dimension 5. It is easily seen as above that a 5-dimensional principally polarized abelian variety with an action of $\mathrm{SL}_2(\mathbb{F}_{11})$ cannot be a Jacobian or a product of Jacobians (see also [Z] for a somewhat analogous, though more sophisticated, proof).

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